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Algebraic Bethe ansatz for the supersymmetric $t-J$ model with reflecting boundary conditions

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Abstract

In the framework of the graded quantum inverse scattering method (QISM), we obtain the eigenvalues and eigenvectors of the supersymmetric $t-J$ model with reflecting boundary conditions in FFB background. The corresponding Bethe ansatz equations are obtained.

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I Introduction

It is believed that the strongly correlated electronic systems are important in studying the high-T_c superconductivity. An appropriate starting point is the $t-J$ model which was proposed by Anderson *et al*.[1,2] The Hamiltonian includes the near-neighbour hopping (t) and antiferromagnetic exchange (J).

$$H=\sum_{j=1}^{L}\left\{-t\mathcal{P}\sum_{\sigma=\pm1}(c_{j,\sigma}^{\dagger}c_{j+1,\sigma}+H.c.)\mathcal{P}+J(\mathbf{S}_j\mathbf{S}_{j+1}-\frac{1}{4}n_n n_{j+1})\right\}. \tag{1}$$

Essler and Korepin *et al* [3] shown that this model is supersymmetric, and the one-dimensional Hamiltonian can be obtained from the transfer matrix of the two-dimensional supersymmetric exactly solvable lattice model. They used the graded QISM and obtained the eigenvalues and eigenvectors for the supersymmetric $t-J$ model with periodic boundary conditions in three different backgrounds.

We know that the exactly solvable models are generally solved by imposing periodic boundary conditions. Recently, solvable models with reflecting boundary conditions have been attracting a great deal of interests.[4,5,6,7] The Hamiltonian includes non-trivial boundary terms which are determined by the boundary K matrices. In the present paper, we will use the algebraic Bethe ansatz method to solve the eigenvalues and eigenvectors problems of the supersymmetric $t-J$ model with reflecting boundary conditions in the famework of the graded QISM (FFB grading), and the Bethe ansatz equations are also obtained.

The paper is organized as follows: In section II, we will introduce the model and the notations. In section III, we will prove the integrability of model with reflecting boundary conditions in the graded sense. The general solution of the reflection equation is also presented in this section. In section IV, we use the algebraic Bethe ansatz method to obtain the eigenvalues and eigenvectors of the supersymmetric $t-J$ model. Section V includes a brief summary and some discussions.

II Description of the model

We first give a brief review of the graded version of the QISM. For convenient we take the natations used by Essler and Korepin.[3] And what we consider in this paper is the FFB grading, that is the grading is fermionic, fermionic and bosonic. In terms of the Grassmann parities this means that $\epsilon_1=\epsilon_2=1$ and $\epsilon_3=0$. The R is defined as

$$\hat{R}(\lambda)=b(\lambda)I+a(\lambda)\Pi, \tag{2}$$

where

$$a(\lambda)=\frac{\lambda}{\lambda+i}, b(\lambda)=\frac{i}{\lambda+i}. \tag{3}$$

And the identity operator is given by $I_{a_1a_2}^{b_1b_2} = \delta_{a_1b_1}\delta_{a_2b_2}$, the matrix Π permutes the individual linear spaces in the tensor product space,

$$\Pi_{a_1a_2}^{b_1b_2} = \delta_{a_1b_2}\delta_{a_2b_1}(-1)^{\epsilon_{b_1}\epsilon_{b_2}}. \tag{4}$$

Explicitly, we can write the R-matrix as:

$$\hat{R}(\lambda) = \begin{pmatrix} b(\lambda)-a(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & b(\lambda) & 0 & -a(\lambda) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & b(\lambda) & 0 & 0 & 0 & a(\lambda) & 0 & 0 \\ 0 & -a(\lambda) & 0 & b(\lambda) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & b(\lambda)-a(\lambda) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & b(\lambda) & 0 & a(\lambda) & 0 \\ 0 & 0 & a(\lambda) & 0 & 0 & 0 & b(\lambda) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a(\lambda) & 0 & b(\lambda) & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{5}$$

As is well known the Yang-Baxter[8,9] relation plays a key role for integrable models with periodic boundary conditions which takes the form

$$\hat{R}(\lambda-\mu)L_n(\lambda)\otimes L_n(\mu) = L_n(\mu)\otimes L_n(\lambda)\hat{R}(\lambda-\mu), \tag{6}$$

where the tensor product is in the graded sense

$$(F\otimes G)_{ac}^{bd} = F_{ab}G_{cd}(-1)^{\epsilon_c(\epsilon_a+\epsilon_b)}. \tag{7}$$

The n means the n-th quantum space which is standard in QISM.[10] We can also write the Yang-Baxter relation explicitly as

$$\begin{aligned} & \hat{R}(\lambda-\mu)_{a_1a_2}^{c_1c_2} L_n(\lambda)_{c_1\alpha_n}^{b_1\gamma_n} L_n(\mu)_{c_2\gamma_n}^{b_2\beta_n}(-1)^{\epsilon_{c_2}(\epsilon_{c_1}+\epsilon_{b_1})} \\ = \; & L_n(\mu)_{a_1\alpha_n}^{c_1\gamma_n} L_n(\lambda)_{a_2\gamma_n}^{c_2\beta_n}(-1)^{\epsilon_{a_2}(\epsilon_{a_1}+\epsilon_{c_1})}\hat{R}(\lambda-\mu)_{c_1c_2}^{b_1b_2}. \end{aligned} \tag{8}$$

The L operator can be constructed from the R-matrix

$$L_n(\lambda)_{a\alpha}^{b\beta} = \Pi_{a\alpha}^{c\gamma}\hat{R}(\lambda)_{c\gamma}^{b\beta} = [b(\lambda)\Pi + a(\lambda)I]_{a\alpha}^{b\beta}. \tag{9}$$

So, the L operator is of the form

$$L_n(\lambda) = \begin{pmatrix} a(\lambda)-b(\lambda)e_n^{11} & -b(\lambda)e_n^{21} & b(\lambda)e_n^{31} \\ -b(\lambda)e_n^{12} & a(\lambda-b(\lambda)e_n^{22} & b(\lambda)e_n^{32} \\ b(\lambda)e_n^{13} & b(\lambda)e_n^{23} & a(\lambda)+b(\lambda)e_n^{33} \end{pmatrix}, \tag{10}$$

where e_n^{ab} are quantum operators acting in the n-th quantum space with matrix representation $(e_n^{ab})_{\alpha\beta} = \delta_{a\alpha}\delta_{b\beta}$. The monodromy matrix $T_L(\lambda)$ defined as the matrix product over the L operators on all site of the lattice,

$$T_L(\lambda) = L_L(\lambda)L_{L-1}(\lambda)\cdots L_1(\lambda), \tag{11}$$

where the tensor product is still in the graded sense, and we will not point it out in the following.

$$\begin{aligned}
&\{[T_L(\lambda)]^{ab}\}_{\substack{\alpha_1\cdots\alpha_L \\ \beta_1\cdots\beta_L}} \\
&= L_L(\lambda)_{a\alpha_L}^{c_L\beta_L} L_{L-1}(\lambda)_{c_L\alpha_{L-1}}^{c_{L-1}\beta_{L-1}} \cdots L_1(\lambda)_{c_2\alpha_1}^{b\beta_1}(-1)^{\sum_{j=2}^{L}(\epsilon_{\alpha_j}+\epsilon_{\beta_j})\sum_{i=1}^{j-1}\epsilon_{\alpha_i}}
\end{aligned} \quad (12)$$

By repeatedly using the Yang-Baxter relation (6), one can prove easily that the monodromy matrix also satisfy the Yang-Baxter relation

$$\hat{R}(\lambda-\mu)[T_L(\lambda)\otimes T_L(\mu)] = T_L(\mu)\otimes T_L(\lambda)\hat{R}(\lambda-\mu), \quad (13)$$

The transfer matrix $\tau_{peri}(\lambda)$ of this model is defined as the supertrace of the monodromy matrix in the auxiliary space. It is defined as the following in the general case

$$\tau_{peri}(\lambda) = str[T_L(\lambda)] = \sum(-1)^{\epsilon_a}[T_L(\lambda)]^{aa}. \quad (14)$$

For the case considered in this paper, if we represent

$$T_L(\lambda) = \begin{pmatrix} A_{11}(\lambda) & A_{12}(\lambda) & B_1(\lambda) \\ A_{21}(\lambda) & A_{22}(\lambda) & B_2(\lambda) \\ C_1(\lambda) & C_2(\lambda) & D(\lambda) \end{pmatrix}. \quad (15)$$

The transfer matrix is then given as

$$\tau(\lambda)_{peri} = -A_{11}(\lambda) - A_{22}(\lambda) + D(\lambda). \quad (16)$$

As a consequence of the Yang-Baxter relation (13), we can prove that the transfer matrix commute with each other for different spectrum parameters.

$$[\tau_{peri}(\lambda), \tau_{peri}(\mu)] = 0 \quad (17)$$

It has been proved that the Hamiltonian obtained by taking the first logarithmic derivative at zero spectral parameter

$$H_{(2)} = -i\frac{d\ln[\tau(\lambda)]}{d\lambda}|_{\lambda=0} = -\sum_{k=1}^{L}\Pi^{k,k+1} \quad (18)$$

is equivalent to the Hamiltonian of the supersymmtric $t-J$ model.[3] Here we have omitted the indentities.

What mentioned above is for periodic boundary conditions. We will study the case of the reflecting boundary conditions for the supersymmetric $t-J$ model. For convenience,

we change the braided R-matrix $\hat{R}$ (2,5) to the non-braided R-matrix

$$
\begin{aligned}
R(\lambda) &= b(\lambda)\Pi + a(\lambda)I \\
&= \begin{pmatrix}
a(\lambda)-b(\lambda) & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & a(\lambda) & 0 & -b(\lambda) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & a(\lambda) & 0 & 0 & 0 & b(\lambda) & 0 & 0 \\
0 & -b(\lambda) & 0 & a(\lambda) & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & a(\lambda)-b(\lambda) & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & a(\lambda) & 0 & b(\lambda) & 0 \\
0 & 0 & b(\lambda) & 0 & 0 & 0 & a(\lambda) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & b(\lambda) & 0 & a(\lambda) & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\end{aligned}
\tag{19}
$$

So, we change the Yang-Baxter relation (13) as:

$$
R(\lambda-\mu)T_1(\lambda)T_2(\mu) = T_2(\mu)T_1(\lambda)R(\lambda-\mu) \tag{20}
$$

Here $1,2$ mean the auxiliary space. The definition for monodromy matrix T remain the same as before.

III Integrability of the supersymmetric $t-J$ model with reflecting boundary conditions

As we know the Yang-Baxter relation is enough to prove the integrability of the exactly solvable model. For the reflecting boundary conditions, besides the Yang-Baxter relation, we also need the reflection equation and the dual reflection equation to prove the integrability of the solvable model. The reflection equation was first proposed by Cherednik.[11] In order to prove the integrability for reflecting boundary conditions, Sklyanin[4] proposed the dual reflection equation. Generally, the dual reflection equation which depends on the unitarity and cross-unitrarity relations of the R-matrix takes different forms for different models. [7]

For the R-matrix considered in this paper (19), one can prove that the R-matrix satisfy the unitarity relation

$$
R_{12}(\lambda)R_{21}(-\lambda) = 1, \tag{21}
$$

where $R_{21} = \Pi R_{12}\Pi$. We can also find that this R-matrix has a symmetry $R_{21} = R_{12}$.

We define the super-transposition st as:

$$
(A^{st})_{ij} = A_{ji}(-1)^{(\epsilon_i+1)\epsilon_j}. \tag{22}
$$

For the case considered in this paper $\epsilon_1 = \epsilon_2 = 1, \epsilon_3 = 0$, we can rewrite the above relation explicitly as:

$$\begin{pmatrix} A_{11} & A_{12} & B_1 \\ A_{21} & A_{22} & B_2 \\ C_1 & C_2 & D \end{pmatrix}^{st} = \begin{pmatrix} A_{11} & A_{21} & C_1 \\ A_{12} & A_{22} & C_2 \\ -B_1 & -B_2 & D \end{pmatrix} \tag{23}$$

Here for convenience, we also define the inverse of the super-transposition $\bar{st}$ as $\{A^{st}\}^{\bar{st}} = A$.

Considering the R-matrix presented above, we find that the R-matrix satisfy the following cross-unitarity relation

$$\begin{aligned} R_{12}^{st_1}(i-\lambda)R_{21}^{st_1}(\lambda) &= \rho(\lambda), \\ \rho(\lambda) &= \frac{(i-\lambda)\lambda}{(\lambda+i)(2i-\lambda)}, \end{aligned} \tag{24}$$

here st_1 means taking super-transposition in the first space.

Next, we introduce the graded version of the reflection equation as:

$$\begin{aligned} & R_{12}(\lambda-\mu)K_1(\lambda)R_{21}(\lambda+\mu)K_2(\mu) \\ = \; & K_2(\mu)R_{12}(\lambda+\mu)K_1(\lambda)R_{21}(\lambda-\mu). \end{aligned} \tag{25}$$

It can also be rewritten as:

$$\begin{aligned} & R(\lambda-\mu)_{a_1a_2}^{b_1b_2}K(\lambda)_{b_1}^{c_1}R(\lambda+\mu)_{b_2c_1}^{c_2d_1}K(\mu)_{c_2}^{d_2}(-1)^{(\epsilon_{b_1}+\epsilon_{c_1})\epsilon_{b_2}} \\ = \; & K(\mu)_{a_2}^{b_2}R(\lambda+\mu)_{a_1b_2}^{b_1c_2}K(\lambda)_{b_1}^{c_1}R(\lambda-\mu)_{c_2c_1}^{d_2d_1}(-1)^{(\epsilon_{b_1}+\epsilon_{c_1})\epsilon_{c_2}}. \end{aligned} \tag{26}$$

Here the refleting K is the solution of the reflection equation. We will just consider the diagonal K matrix in this paper, so we suppose

$$K(\lambda)_a^b = \delta_{ab}k_a(\lambda). \tag{27}$$

Subsititing this condition into the reflection equation (26), we find the only non-trivial relation is

$$\begin{aligned} & R(\lambda-\mu)_{a_1a_2}^{a_1a_2}R(\lambda+\mu)_{a_2a_1}^{a_1a_2}k(\lambda)_{a_1}k(\mu)_{a_1} + R(\lambda-\mu)_{a_1a_2}^{a_2a_1}R(\lambda+\mu)_{a_1a_2}^{a_1a_2}k(\lambda)_{a_2}k(\mu)_{a_1} \\ = \; & R(\lambda+\mu)_{a_1a_2}^{a_1a_2}R(\lambda-\mu)_{a_2a_1}^{a_1a_2}k(\mu)_{a_2}k(\lambda)_{a_1} + R(\lambda+\mu)_{a_1a_2}^{a_2a_1}R(\lambda-\mu)_{a_1a_2}^{a_2a_1}k(\mu)_{a_2}k(\lambda)_{a_2}. \end{aligned} \tag{28}$$

Solving this relation, we can find two different types of solution to the graded reflection equation

$$K_I(\lambda) = \begin{pmatrix} \xi+\lambda & & \\ & \xi+\lambda & \\ & & \xi-\lambda \end{pmatrix}, \tag{29}$$

$$K_{II}(\lambda) = \begin{pmatrix} \xi+\lambda & & \\ & \xi-\lambda & \\ & & \xi-\lambda \end{pmatrix}, \tag{30}$$

where ξ is an arbitrary parameter. According to the form of the cross-unitarity relation of the R-matrix (24), we propose the graded dual reflection equation take the form

$$\begin{aligned} & R_{12}(\mu-\lambda)K_1^+(\lambda)R_{21}(i-\lambda-\mu)K_2^+(\mu) \\ = \ & K_2^+(\mu)R_{12}(i-\lambda-\mu)K_1^+(\lambda)R_{21}(\mu-\lambda). \end{aligned} \tag{31}$$

One can find that there is an isomorphism between the reflection equation (25) and dual reflection equation (31): Given a solution of the reflection equation (25), we can also find a solution of the dual reflection equation (31). But in the sense of the commuting transfer matrix, the reflection equation and the dual reflection equation are independent of each other. We have two types of the solutions of the dual reflection equation

$$K_I^+(\lambda) = \begin{pmatrix} \xi^+-\lambda & & \\ & \xi^+-\lambda & \\ & & \xi^+-i+\lambda \end{pmatrix}, \tag{32}$$

$$K_{II}^+(\lambda) = \begin{pmatrix} \xi^+-\lambda & & \\ & \xi^+-i+\lambda & \\ & & \xi^+-i+\lambda \end{pmatrix}, \tag{33}$$

here ξ^+ is an arbitrary parameter which is independent of ξ.

Note here that there are two different types of solutions K and K^+, respectively, and we know that K and K^+ are independent of each other in the sense of the transfer matrix, so there are four different transfer matrix altogether corresponding to those K and K^+: $\{K_I^+, K_I\}$; $\{K_I^+, K_{II}\}$; $\{K_{II}^+, K_I\}$; $\{K_{II}^+, K_{II}\}$.

Following the method of Sklyanin,[4] we define the double-row monodromy matrix for the case of reflecting boundary conditions

$$\mathcal{T}(\lambda) = T(\lambda)K(\lambda)T^{-1}(-\lambda). \tag{34}$$

Using the Yang-Baxter relation (20), one can prove easily that this double-row monodromy matrix also satisfy the reflection equation (25)

$$\begin{aligned} & R_{12}(\lambda-\mu)\mathcal{T}_1(\lambda)R_{21}(\lambda+\mu)\mathcal{T}_2(\mu) \\ = \ & \mathcal{T}_2(\mu)R_{12}(\lambda+\mu)\mathcal{T}_1(\lambda)R_{21}(\lambda-\mu). \end{aligned} \tag{35}$$

We thus define the tranfer matrix with open boundary conditions as:

$$t(\lambda) = str K^+(\lambda)\mathcal{T}(\lambda), \tag{36}$$

As before *str* means super trace. Next, we will prove that the defined transfer matrices with different spectral parameters commute with each other. We generally in this sense mean the model is integrable.

We first take super transposition in the first space.

$$\begin{aligned} t(\lambda)t(\mu) &= str_1 K_1^+(\lambda)\mathcal{T}_1(\lambda) str_2 K_2^+(\mu)\mathcal{T}_2(\mu) \\ &= str_{12} K_1^+(\lambda)^{st_1} K_2^+(\mu)\mathcal{T}_1^{st_1}(\lambda)\mathcal{T}_2(\mu), \end{aligned}$$

Now we insert the cross-unitrarity relation (24) of the R-matrix, and take inverse of super transposition in the first space, we have

$$\begin{aligned} \cdots &= \frac{1}{\rho(\lambda+\mu)} str_{12} K_1^+(\lambda)^{st_1} K_2^+(\mu) R_{12}^{st_1}(i-\lambda-\mu) R_{21}^{st_1}(\lambda+\mu)\mathcal{T}_1^{st_1}(\lambda)\mathcal{T}_2(\mu), \\ &= \frac{1}{\rho(\lambda+\mu)} str_{12}\{K_1^+(\lambda)^{st_1} K_2^+(\mu) R_{12}^{st_1}(i-\lambda-\mu)\}^{\bar{st}_1} \\ &\quad \{\mathcal{T}_1(\lambda) R_{21}(\lambda+\mu)\mathcal{T}_2(\mu)\}, \\ &= \frac{1}{\rho(\lambda+\mu)} str_{12}\{K_2^+(\mu) R_{12}(i-\lambda-\mu) K_1^+(\lambda)\}\{\mathcal{T}_1(\lambda) R_{21}(\lambda+\mu)\mathcal{T}_2(\mu)\}. \end{aligned}$$

Insert the untarity relation of the R-matrix (21), and use the RE (35) and the dual RE (31), we have

$$\begin{aligned} \cdots &= \frac{1}{\rho(\lambda+\mu)} str_{12}\{K_2^+(\mu) R_{12}(i-\lambda-\mu) K_1^+(\lambda) R_{21}(\mu-\lambda)\} \\ &\quad \{R_{12}(\lambda-\mu)\mathcal{T}_1(\lambda) R_{21}(\lambda+\mu)\mathcal{T}_2(\mu)\}. \\ &= \frac{1}{\rho(\lambda+\mu)} str_{12}\{R_{12}(\mu-\lambda) K_1^+(\lambda) R_{21}(i-\mu-\lambda) K_2^+(\mu)\} \\ &\quad \{\mathcal{T}_2(\mu) R_{12}(\lambda+\mu)\mathcal{T}_1(\lambda) R_{21}(\lambda-\mu)\}. \end{aligned} \tag{37}$$

Applying almost the same procedure as before, use again the unitarity relation (21) and the cross-unitarity relation (24), we have

$$\begin{aligned} \cdots &= \frac{1}{\rho(\lambda+\mu)} str_{12}\{K_1^+(\lambda) R_{21}(i-\mu-\lambda) K_2^+(\mu)\}\{\mathcal{T}_2(\mu) R_{12}(\lambda+\mu)\mathcal{T}_1(\lambda)\}. \\ &= \frac{1}{\rho(\lambda+\mu)} str_{12}\{R_{21}^{st_1}(i-\mu-\lambda) K_1^+(\lambda)^{st_1} K_2^+(\mu)\} \\ &\quad \{\mathcal{T}_2(\mu)\mathcal{T}_1^{st_1}(\lambda) R_{12}^{st_1}(\lambda+\mu)\}. \\ &= str_2 K_2^+(\mu)\mathcal{T}_2(\mu) str_1 K_1^+(\lambda)\mathcal{T}_1(\lambda) \\ &= t(\mu)t(\lambda). \end{aligned} \tag{38}$$

Thus we have proved that the transfer matrix constitute a commuting family which gives an infinite set of conserved quantities.

Corresponding to this transfer matrix, we can also obtain the Hamiltonian:

$$H_{(2)}^{Bound.} = -\frac{i}{2}\frac{d\ln[t(\lambda)]}{d\lambda}|_{\lambda=0} = -\sum_{k=1}^{L-1}\Pi^{k,k+1} - \frac{i}{2}K_1'(0) - \frac{str_1 K_1^+(0)\Pi^{L,1}}{str K^+(0)}.$$

The boundary terms are determined by the reflecting K matrices.

IV Algebraic Bethe ansatz method

IV.1 Transfer matrix and the vacuum state

According to the definition of the monodromy matrix (11), we can write the inverse of the monodromy matrix as:

$$T^{-1}(-\lambda) \;=\; L_1^{-1}(-\lambda)L_2^{-1}(-\lambda)\cdots L_L^{-1}(-\lambda) \tag{39}$$

With the help of the definition relation (34), we can rewrite the double-row monodromy matrix explicitly as:

$$\begin{aligned}
\mathcal{T}(\lambda) &= T(\lambda)K(\lambda)T^{-1}(-\lambda) \\
&= \begin{pmatrix} A_{11}(\lambda) & A_{12}(\lambda) & B_1(\lambda) \\ A_{21}(\lambda) & A_{22}(\lambda) & B_2(\lambda) \\ C_1(\lambda) & C_2(\lambda) & D(\lambda) \end{pmatrix} \times \begin{pmatrix} k_1(\lambda) & 0 & 0 \\ 0 & k_2(\lambda) & 0 \\ 0 & 0 & k_3(\lambda) \end{pmatrix} \\
&\times \begin{pmatrix} \bar{A}_{11}(-\lambda) & \bar{A}_{12}(-\lambda) & \bar{B}_1(-\lambda) \\ \bar{A}_{21}(-\lambda) & \bar{A}_{22}(-\lambda) & \bar{B}_2(-\lambda) \\ \bar{C}_1(-\lambda) & \bar{C}_2(-\lambda) & \bar{D}(-\lambda) \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{A}_{11}(\lambda) & \mathcal{A}_{12}(\lambda) & \mathcal{B}_1(\lambda) \\ \mathcal{A}_{21}(\lambda) & \mathcal{A}_{22}(\lambda) & \mathcal{B}_2(\lambda) \\ \mathcal{C}_1(\lambda) & \mathcal{C}_2(\lambda) & \mathcal{D}(\lambda) \end{pmatrix}.
\end{aligned} \tag{40}$$

For the periodic boundary conditions, Essler and Korepin [3]choose the reference state in the k-th quantum space and the vacuum $|0>$ as:

$$|0>_n = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, |0> = \otimes_{k=1}^{L}|0>_k \;. \tag{41}$$

What we study in this paper is the the case of the reflecting boundary conditions, we assume the vacuum state remain the same as the case of periodic boundary conditions. That means the above state $|0>$ is still the vacuum state for the reflecting boundary conditions. According to the definition of the monodromy matrix $T(\lambda)$ and the inverse of the monodromy matrix $T^{-1}(\lambda)$, we have the following results:

$$\begin{aligned}
T(\lambda)|0> &= \begin{pmatrix} [a(\lambda)]^L & 0 & 0 \\ 0 & [a(\lambda)]^L & 0 \\ C_1(\lambda) & C_2(\lambda) & 1 \end{pmatrix}|0> \\
T^{-1}(-\lambda)|0> &= \begin{pmatrix} [a(\lambda)]^L & 0 & 0 \\ 0 & [a(\lambda)]^L & 0 \\ \bar{C}_1(-\lambda) & \bar{C}_2(-\lambda) & 1 \end{pmatrix}|0> .
\end{aligned} \tag{42}$$

Now let us see the values of the double-row monodromy matrix $\mathcal{T}$ acting on the vacuum state. One can obtain easily

$$\mathcal{D}(\lambda)|0>= k_3(\lambda)D(\lambda)\bar{D}(-\lambda)|0>= k_3(\lambda)|0>, \tag{43}$$

$$\mathcal{B}_1(\lambda)|0>=0, \quad \mathcal{B}_2(\lambda)|0>=0, \tag{44}$$

$$\mathcal{C}_1(\lambda)|0>\neq 0, \quad \mathcal{C}_2(\lambda)|0>\neq 0. \tag{45}$$

It is non-trivial for the other elements,

$$\begin{aligned}
\mathcal{A}_{12}(\lambda)|0> &= k_3(\lambda)B_1(\lambda)\bar{C}_2(-\lambda)|0>, \\
\mathcal{A}_{21}(\lambda)|0> &= k_3(\lambda)B_2(\lambda)\bar{C}_1(-\lambda)|0>, && (46)\\
\mathcal{A}_{22}(\lambda)|0> &= [k_2(\lambda)A_{22}(\lambda)\bar{A}_{22}(-\lambda)+k_3(\lambda)B_2(\lambda)\bar{C}_2(-\lambda)]|0>, && (47)\\
\mathcal{A}_{11}(\lambda)|0> &= [k_1(\lambda)A_{11}(\lambda)\bar{A}_{11}(-\lambda)+k_3(\lambda)B_1(\lambda)\bar{C}_1(-\lambda)]|0>. && (48)
\end{aligned}$$

In order to obtain the results of the above relations, we should use the graded Yang-Baxter relation. From relation (20), we can find the following explicit relation

$$\begin{aligned}
&[T^{-1}(-\lambda)]^{b_2}_{a_2}R(2\lambda)^{b_1c_2}_{a_1b_2}T(\lambda)^{c_1}_{b_1}(-1)^{(\epsilon_{b_1}+\epsilon_{c_1})\epsilon_{c_2}} \\
= \ &T(\lambda)^{b_1}_{a_1}R(2\lambda)^{c_1b_2}_{b_1a_2}[T^{-1}(-\lambda)]^{c_2}_{b_2}(-1)^{(\epsilon_{a_1}+\epsilon_{b_1})\epsilon_{a_2}}.
\end{aligned} \tag{49}$$

Acting the two sides of this relation on the vacuum state, and taking special values for the indecies, for cases $a_1=1,a_2=3,c_1=3,c_2=2$ and $a_1=2,a_2=3,c_1=3,c_2=1$, with the help of relation (42), we have the results:

$$\begin{aligned}
\mathcal{A}_{12}(\lambda)|0> &= 0, \\
\mathcal{A}_{21}(\lambda)|0> &= 0.
\end{aligned} \tag{50}$$

For case $a_1=2,a_2=3,c_1=3,c_2=2$, we have

$$B_2(\lambda)\bar{C}_2(-\lambda)|0>= b(2\lambda)\bar{D}(-\lambda)D(\lambda)|0> -b(2\lambda)A_{22}(\lambda)\bar{A}_{22}(-\lambda)|0>. \tag{51}$$

Substituting this relation into relation (47), we find

$$\mathcal{A}_{22}(\lambda)|0>=\{[k_2(\lambda)-k_3(\lambda)b(2\lambda)]a^{2L}(\lambda)+b(2\lambda)k_3(\lambda)\}|0>, \tag{52}$$

here for convenience, we introduce a transformation

$$\mathcal{A}_{22}(\lambda)=\tilde{\mathcal{A}}_{22}(\lambda)+b(2\lambda)\mathcal{D}(\lambda). \tag{53}$$

Thus we can find the value of the element $\tilde{\mathcal{A}}_{22}$ acting on vacuum state

$$\tilde{\mathcal{A}}_{22}(\lambda)|0>=[k_2(\lambda)-k_3(\lambda)b(2\lambda)]a^{2L}(\lambda)|0>. \tag{54}$$

The above transformation is very important in the later algebraic Bethe ansatz method. Instead of $\mathcal{A}_{22}(\lambda)$, we use $\tilde{\mathcal{A}}_{22}(\lambda)$ acting on the assumed eigenvectors, so we find that there are only one wanted term which is necessary for the algebraic Bethe anstaz method. Similarly, we have the relation

$$B_1(\lambda)\bar{C}_1(-\lambda)|0>= b(2\lambda)\bar{D}(-\lambda)D(\lambda)|0> -b(2\lambda)A_{11}(\lambda)\bar{A}_{11}(-\lambda)|0> . \tag{55}$$

Introduce similar transformation

$$\mathcal{A}_{11}(\lambda) = \tilde{\mathcal{A}}_{11}(\lambda) + b(2\lambda)\mathcal{D}(\lambda). \tag{56}$$

We have

$$\tilde{\mathcal{A}}_{11}(\lambda)|0>= [k_1(\lambda) - k_3(\lambda)b(2\lambda)]a^{2L}(\lambda)|0> . \tag{57}$$

We summarize the above results

$$\begin{aligned} \tilde{\mathcal{A}}_{11}(\lambda)|0> &= W_1(\lambda)a^{2L}(\lambda)|0>, \\ \tilde{\mathcal{A}}_{22}(\lambda)|0> &= W_2(\lambda)a^{2L}(\lambda)|0>, \\ \mathcal{D}(\lambda)|0> &= U_3^I(\lambda)|0> \end{aligned} \tag{58}$$

Corresponding to two different types of solutions K of the reflection equation. $W_j(\lambda), j = 1, 2$, and $U_3(\lambda)$ take following values

For $K_I(\lambda)$:

$$\begin{aligned} W_1(\lambda) &= \frac{2\lambda(\lambda+\xi+i)}{2\lambda+i} \\ W_2(\lambda) &= \frac{2\lambda(\lambda+\xi+i)}{2\lambda+i} \\ U_3(\lambda) &= (\xi-\lambda) \end{aligned} \tag{59}$$

For $K_{II}(\lambda)$:

$$\begin{aligned} W_1(\lambda) &= \frac{2\lambda(\lambda+\xi+i)}{2\lambda+i} \\ W_2(\lambda) &= \frac{2\lambda(\xi-\lambda)}{2\lambda+i} \\ U_3(\lambda) &= (\xi-\lambda) \end{aligned} \tag{60}$$

Considering the transformation (53,56) and definition of the transfer matrix with reflecting boundary conditions, we can rewrite the transfer matrix as:

$$\begin{aligned} t(\lambda) &= strK^+(\lambda)\mathcal{T}(\lambda) = -k_1^+(\lambda)\mathcal{A}_{11}(\lambda) - k_2^+(\lambda)\mathcal{A}_{22}(\lambda) + k_3^+(\lambda)\mathcal{D}(\lambda) \\ &= -W_1^+(\lambda)\tilde{\mathcal{A}}_{11}(\lambda) - W_2^+(\lambda)\tilde{\mathcal{A}}_{22}(\lambda) + U_3^+(\lambda)\mathcal{D}(\lambda) \end{aligned} \tag{61}$$

Here $W_j^+, j = 1, 2$ and U_3^+ take the following form

For $K_I^+(\lambda)$:

$$\begin{aligned} W_1^+(\lambda) &= \xi^+ - \lambda, \\ W_2^+(\lambda) &= \xi^+ - \lambda, \\ U_3^+(\lambda) &= \frac{(2\lambda - i)(\xi^+ + \lambda + i)}{2\lambda + i}, \end{aligned} \tag{62}$$

For $K_{II}^+(\lambda)$:

$$\begin{aligned} W_1^+(\lambda) &= \xi^+ - \lambda, \\ W_2^+(\lambda) &= \xi^+ - i + \lambda, \\ U_3^+(\lambda) &= \frac{(2\lambda - i)(\xi^+ + \lambda)}{2\lambda + i}. \end{aligned} \tag{63}$$

IV.2 Commutation relations and the first step of the nested algebraic Bethe ansatz method

For the algebraic Bethe ansatz method, we should obtain the commutation relations between the elements of $\mathcal{T}$. In the case of reflecting boundary condition, instead of the Yang-Baxter relation, we need the reflection equation (35) to obtain the necessary commutation relations. First of all, we introduce a transformation

$$\mathcal{A}_{ab}(\lambda) = \tilde{\mathcal{A}}_{ab}(\lambda) + \delta_{ab} b(2\lambda)\mathcal{D}(\lambda) \tag{64}$$

which is consistent with the former transformations (53, 56). Next We intend to find the commuation relations between $\tilde{\mathcal{A}}_{aa}$, $\mathcal{D}$ and $\mathcal{C}_b$. The commuation relation between $\mathcal{A}_{aa}$ and $\mathcal{C}_b$ is not necessary, because there will appear two or more wanted terms which can not be handle for the algebraic Bethe ansatz method. For convenience, we write the reflection equation explicitly as:

$$\begin{aligned} &R(\lambda - \mu)_{a_1 a_2}^{b_1 b_2} \mathcal{T}(\lambda)_{b_1}^{c_1} R_{21}(\lambda + \mu)_{c_1 b_2}^{d_1 c_2} \mathcal{T}(\mu)_{c_2}^{d_2} (-1)^{(\epsilon_{b_1} + \epsilon_{c_1})\epsilon_{b_2}} \\ &= \mathcal{T}(\mu)_{a_2}^{b_2} R(\lambda + \mu)_{a_1 b_2}^{b_1 c_2} \mathcal{T}(\lambda)_{b_1}^{c_1} R_{21}(\lambda - \mu)_{c_1 c_2}^{d_1 d_2} (-1)^{(\epsilon_{b_1} + \epsilon_{c_1})\epsilon_{c_2}} \end{aligned} \tag{65}$$

Take special values for the indecies of this reflection equation, for case $a_1 = a_2 = 3, d_1, d_2 \neq 3$, we find

$$\mathcal{C}_{d_1}(\lambda)\mathcal{C}_{d_2}(\mu) = -\mathcal{C}_{c_2}(\mu)\mathcal{C}_{c_1}(\lambda) R(\lambda - \mu)_{c_2 c_1}^{d_2 d_1}. \tag{66}$$

That means that $\mathcal{C}(\lambda)$ and $\mathcal{C}(\mu)$ are commutative up to a scalar. Later we will use this property to construct the eigenvector of the transfer matrix. Note here all indices in the commutation relation take values $1, 2$. For other commutation relations, this is also true

and we will not point it out later. For case $a_1 = a_2 = d_2 = 3, d_1 \neq 3$, and considering the transformation (61), we have the commutation relation between $\mathcal{D}$ and $\mathcal{C}$,

$$\begin{aligned}\mathcal{D}(\lambda)\mathcal{C}_d(\mu) &= \frac{(\lambda+\mu)(\lambda-\mu-i)}{(\lambda+\mu+i)(\lambda-\mu)}\mathcal{C}_d(\mu)\mathcal{D}(\lambda) \\ &+ \frac{2i\mu}{(\lambda-\mu)(2\mu+i)}\mathcal{C}_d(\lambda)\mathcal{D}(\mu) - \frac{i}{\lambda+\mu+i}\mathcal{C}_b(\lambda)\tilde{\mathcal{A}}_{bd}(\mu). \end{aligned} \tag{67}$$

To obtain the commutation relation between $\tilde{\mathcal{A}}$ and $\mathcal{C}$, the calculation is much complicated and tedious. Here we just give a sketch of it. Take indices $a_2 = 3, a_1, d_1, d_2 \neq 3$, we have

$$\begin{aligned} & a(\lambda-\mu)a(\lambda+\mu)\mathcal{A}_{a_1d_1}(\lambda)\mathcal{C}_{d_2}(\mu) - (1-\delta_{a_1d_1})b(\lambda-\mu)a(\lambda+\mu)\mathcal{C}_{d_1}(\lambda)\mathcal{A}_{a_1d_2}(\mu) \\ + \ & \delta_{a_1d_1}\{b(\lambda-\mu)b(\lambda+\mu)\mathcal{D}(\lambda)\mathcal{C}_{d_2}(\mu) - b(\lambda-\mu)R_{21}(\lambda+\mu)^{a_1c_1}_{c_1a_1}\mathcal{C}_{c_1}(\lambda)\mathcal{A}_{c_1d_2}(\mu)\} \\ = \ & -\mathcal{T}(\mu)^3_3 R(\lambda+\mu)^{3a_1}_{a_13}\mathcal{T}(\lambda)^{c_1}_3 R_{21}(\lambda-\mu)^{d_1d_2}_{c_1a_1} \\ + \ & \mathcal{T}(\mu)^{b_2}_3 R(\lambda+\mu)^{b_1c_2}_{a_1b_2}\mathcal{T}(\lambda)^{c_1}_{b_1} R_{21}(\lambda-\mu)^{d_1d_2}_{c_1c_2}. \end{aligned} \tag{68}$$

Subsitute the transformation (61) into this relation and consider it for three cases: Case I: $a_1 \neq d_1, d_1 = d_2 or d_1 \neq d_2$; Case II: $a_1 = d_1 = d_2$; Case III: $a_1 = d_1 \neq d_2$. The results of the above relation can be calculated out. However, it is still too complicated to be handled for the algebraic Bethe ansatz method. Fortunately, we can summarize all of those results to a much concise relation:

$$\begin{aligned} & \tilde{\mathcal{A}}_{a_1d_1}(\lambda)\mathcal{C}_{d_2}(\mu) \\ = \ & \frac{(\lambda-\mu+i)(\lambda+\mu+2i)}{(\lambda-\mu)(\lambda+\mu+i)} r_{12}(\lambda+\mu+i)^{c_1b_2}_{a_1c_2} r_{21}(\lambda-\mu)^{d_1d_2}_{b_1b_2}\mathcal{C}_{c_2}(\mu)\tilde{\mathcal{A}}_{c_1b_1}(\lambda) \\ + \ & \frac{2i(\lambda+i)}{(\lambda-\mu)(2\lambda+i)} r(2\lambda+i)^{b_2d_1}_{a_1b_1}\mathcal{C}_{b_1}(\lambda)\tilde{\mathcal{A}}_{b_2d_2}(\mu) \\ - \ & \frac{4i\mu(\lambda+i)}{(2\lambda+i)(2\mu+i)(\lambda+\mu+2i)} r(2\lambda+i)^{d_2d_1}_{a_1b_2}\mathcal{C}_{b_2}(\lambda)\mathcal{D}(\mu). \end{aligned} \tag{69}$$

Where the elements of the r-matrix are defined as the elements of the original R matrix for the case all of its indices just take values 1,2.

$$r(\lambda)^{bd}_{ac} = a(\lambda)\delta_{ab}\delta_{cd} - b(\lambda)\delta_{ac}\delta_{bd} = a(\lambda)I + b(\lambda)\Pi^{(1)}, \tag{70}$$

where $\Pi^{(1)}$ is the 4×4 permutation matrix corresponding to the grading $\epsilon_1 = \epsilon_2 = 1$. We can write the r-matrix as:

$$r(\lambda) = \begin{pmatrix} a(\lambda)-b(\lambda) & & & \\ & a(\lambda) & -b(\lambda) & \\ & -b(\lambda) & a(\lambda) & \\ & & & a(\lambda)-b(\lambda) \end{pmatrix} \tag{71}$$

Similar as the periodic boundary condition case, we construct a set of the eigenvectors of the transfer matrix with reflecting boundary conditions as:

$$\mathcal{C}_{d_1}(\mu_1)\mathcal{C}_{d_2}(\mu_2)\cdots\mathcal{C}_{d_n}(\mu_n)|0>F^{d_1\cdots d_n}. \tag{72}$$

Here $F^{d_1\cdots d_n}$ is a function of the spectral parameters μ_j. This technique is standard for the algebraic Bethe ansatz method. Acting the transfer matrix on this eigenvectors, we should find the eigenvalues $\Lambda(\lambda)$ of the transfer matrix $t(\lambda)$ and a set of Bethe ansatz equations. Act first $\mathcal{D}$ on the eigenvector defined above, use next the commutation relation (67), consider the value of $\mathcal{D}$ acting on the vacuum state (58), we have

$$\begin{aligned} & \mathcal{D}(\lambda)\mathcal{C}_{d_1}(\mu_1)\mathcal{C}_{d_2}(\mu_2)\cdots\mathcal{C}_{d_n}(\mu_n)|0>F^{d_1\cdots d_n} \\ = \; & U_3(\lambda)\prod_{i=1}^{n}\frac{(\lambda+\mu_i)(\lambda-\mu_i-i)}{(\lambda+\mu_i+i)(\lambda-\mu_i)}\mathcal{C}_{d_1}(\mu_1)\mathcal{C}_{d_2}(\mu_2)\cdots\mathcal{C}_{d_n}(\mu_n)|0>F^{d_1\cdots d_n}+u.t., \end{aligned} \tag{73}$$

where we omited the unwanted terms $u.t.$.

Then we act $\tilde{\mathcal{A}}_{aa}(\lambda)$ on the assumed eigenvector, using repeatedly the commutation relations (69), we have

$$\begin{aligned} & \tilde{\mathcal{A}}_{aa}(\lambda)\mathcal{C}_{d_1}(\mu_1)\mathcal{C}_{d_2}(\mu_2)\cdots\mathcal{C}_{d_n}(\mu_n)|0>F^{d_1\cdots d_n} \\ = \; & \prod_{i=1}^{n}\frac{(\lambda-\mu_i+i)(\lambda+\mu_i+2i)}{(\lambda-\mu_i)(\lambda+\mu_i+i)}r_{12}(\lambda+\mu_1+i)_{ac_1}^{a_1e_1}r_{21}(\lambda-\mu_1)_{b_1e_1}^{ad_1} \\ & r_{12}(\lambda+\mu_2+i)_{a_1c_2}^{a_2e_2}r_{21}(\lambda-\mu_2)_{b_2e_2}^{b_1d_2}\cdots r_{12}(\lambda+\mu_n+i)_{a_{n-1}c_n}^{a_ne_n}r_{21}(\lambda-\mu_n)_{b_ne_n}^{b_{n-1}d_n} \\ & \times\mathcal{C}_{c_1}(\mu_1)\cdots\mathcal{C}_{c_n}(\mu_n)\tilde{\mathcal{A}}_{a_nb_n}(\lambda)|0>F^{d_1\cdots d_n}+u.t.. \end{aligned} \tag{74}$$

Summarize relations (50,58), we know

$$\mathcal{A}_{a_nb_n}(\lambda)|0>=\delta_{a_nb_n}W_{a_n}(\lambda)a^{2L}(\lambda)|0>. \tag{75}$$

We can rewite the transfer matrix as:

$$\begin{aligned} t(\lambda) &= -W_1^+(\lambda)\tilde{\mathcal{A}}_{11}(\lambda)-W_2^+(\lambda)\tilde{\mathcal{A}}_{22}(\lambda)+U_3^+(\lambda)\mathcal{D}(\lambda) \\ &= -W_a^+(\lambda)\tilde{\mathcal{A}}_{aa}(\lambda).+U_3^+(\lambda)\mathcal{D}(\lambda) \end{aligned} \tag{76}$$

Thus the eigenvalue of the transfer matrix with reflecting boundary condition is written as:

$$\begin{aligned} & t(\lambda)\mathcal{C}_{d_1}(\mu_1)\mathcal{C}_{d_2}(\mu_2)\cdots\mathcal{C}_{d_n}(\mu_n)|0>F^{d_1\cdots d_n} \\ = \; & U_3^+(\lambda)U_3(\lambda)\prod_{i=1}^{n}\frac{(\lambda+\mu_i)(\lambda-\mu_i-i)}{(\lambda+\mu_i+i)(\lambda-\mu_i)}\mathcal{C}_{d_1}(\mu_1)\cdots\mathcal{C}_{d_n}(\mu_n)|0>F^{d_1\cdots d_n} \\ + \; & a^{2L}(\lambda)\prod_{i=1}^{n}\frac{(\lambda-\mu_i+i)(\lambda+\mu_i+2i)}{(\lambda-\mu_i)(\lambda+\mu_i+i)}\mathcal{C}_{c_1}(\mu_1)\cdots\mathcal{C}_{c_n}(\mu_n)|0>t^{(1)}(\lambda)_{d_1\cdots d_n}^{c_1\cdots c_n}F^{d_1\cdots d_n} \\ + \; & u.t., \end{aligned} \tag{77}$$

where $t^{(1)}(\lambda)$ is the so called nested transfer matrix, and with the help of the relation (74), it can be defined as:

$$
\begin{aligned}
t^{(1)}(\lambda)_{d_1\cdots d_n}^{c_1\cdots c_n} &= -W_a^+(\lambda)\left\{r(\lambda+\mu_1+i)_{ac_1}^{a_1e_1}r(\lambda+\mu_2+i)_{a_1c_2}^{a_2e_2}\cdots r(\lambda+\mu_1+i)_{a_{n-1}c_n}^{a_ne_n}\right\} \\
&\qquad \delta_{a_nb_n}W_{a_n}(\lambda)\left\{r_{21}(\lambda-\mu_n)_{b_ne_n}^{b_{n-1}d_n}\cdots r_{21}(\lambda-\mu_2)_{b_2e_2}^{b_1d_2}r_{21}(\lambda-\mu_1)_{b_1e_1}^{ad_1}\right\}. \quad (78)
\end{aligned}
$$

We find that this nested transfer matrix can be defined as a transfer matrix with reflecting boundary conditions corresponding to the anisotropic case

$$
t^{(1)}(\lambda) = strK^{(1)^+}(\tilde{\lambda})T^{(1)}(\tilde{\lambda},\{\tilde{\mu}_i\})K^{(1)}(\tilde{\lambda})T^{(1)^{-1}}(-\tilde{\lambda},\{\tilde{\mu}_i\}) \quad (79)
$$

with the grading $\epsilon_1=\epsilon_2=1$, where we denote $\tilde{\lambda}=\lambda+\frac{i}{2},\tilde{\xi}=\xi+\frac{i}{2},\tilde{\xi}^+=\xi^+-\frac{i}{2}$, we will also denote $\tilde{\mu}=\mu+\frac{i}{2}$ later. Explicitly we have $K^{(1)}{}_I^+(\tilde{\lambda})=id.$ up to a whole factor $\tilde{\xi}^++i-\tilde{\lambda}$, and

$$
K^{(1)}{}_{II}^+(\tilde{\lambda}) = \begin{pmatrix} W_1^+(\tilde{\lambda}) & \\ & W_2^+(\tilde{\lambda}) \end{pmatrix} = \begin{pmatrix} \tilde{\xi}^++i-\tilde{\lambda} & \\ & \tilde{\xi}^+-i+\tilde{\lambda} \end{pmatrix} \quad (80)
$$

corresponding to K_I^+ and K_{II}^+ respectively. We also have $K_I^{(1)}(\tilde{\lambda})=id.$ up to a whole factor $\frac{(2\tilde{\lambda}-i)(\tilde{\lambda}+\tilde{\xi})}{2\tilde{\lambda}}$, and

$$
K_{II}^{(1)}(\tilde{\lambda}) = \begin{pmatrix} W_1(\tilde{\lambda}) & \\ & W_2(\tilde{\lambda}) \end{pmatrix} = \frac{(2\tilde{\lambda}-i)}{2\tilde{\lambda}}\begin{pmatrix} \tilde{\xi}+\tilde{\lambda} & \\ & \tilde{\xi}-\tilde{\lambda} \end{pmatrix} \quad (81)
$$

corresponding to K_I and K_{II}. The row-to-row monodromy matrix $T^{(1)}(\tilde{\lambda},\{\tilde{\mu}_i\})$ (corresponding to the periodic boundary condition) is defined as:

$$
\begin{aligned}
T_{aa_n}^{(1)}(\tilde{\lambda},\{\tilde{\mu}_i\})_{c_1\cdots c_n}^{e_1\cdots e_n} &= r(\tilde{\lambda}+\tilde{\mu}_1)_{ac_1}^{a_1e_1}r(\tilde{\lambda}+\tilde{\mu}_2)_{a_1c_2}^{a_2e_2}\cdots r(\tilde{\lambda}+\tilde{\mu}_1)_{a_{n-1}c_n}^{a_ne_n} \\
&= L_1^{(1)}(\tilde{\lambda}+\tilde{\mu}_1)L_2^{(1)}(\tilde{\lambda}+\tilde{\mu}_2)\cdots L_n^{(1)}(\tilde{\lambda}+\tilde{\mu}_1). \quad (82)
\end{aligned}
$$

The L-operator takes the form

$$
L_k^{(1)}(\tilde{\lambda}) = \begin{pmatrix} a(\tilde{\lambda})-b(\tilde{\lambda})e_k^{11} & -b(\tilde{\lambda})e_k^{21} \\ -b(\tilde{\lambda})e_k^{12} & a(\tilde{\lambda})-b(\tilde{\lambda})e_k^{22} \end{pmatrix}. \quad (83)
$$

And we also have

$$
\begin{aligned}
T^{(1)^{-1}}(-\tilde{\lambda},\{\tilde{\mu}_i\}) &= r_{21}(\tilde{\lambda}-\tilde{\mu}_n)_{b_ne_n}^{b_{n-1}d_n}\cdots r_{21}(\tilde{\lambda}-\tilde{\mu}_2)_{b_2e_2}^{b_1d_2}r_{21}(\tilde{\lambda}-\tilde{\mu}_1)_{b_1e_1}^{ad_1} \\
&= L_n^{(1)^{-1}}(-\tilde{\lambda}+\tilde{\mu}_n)\cdots L_2^{(1)^{-1}}(-\tilde{\lambda}+\tilde{\mu}_2)L_1^{(1)^{-1}}(-\tilde{\lambda}+\tilde{\mu}_1), \quad (84)
\end{aligned}
$$

where we have used the unitarity relation of the r-matrix $r_{12}(\lambda)r_{21}(-\lambda)=1$.

In this section, We show that the problem to find the eigenvalue of the original transfer matrix $t(\lambda)$ become the problem to find the eignevalue of the nested transfer matrix $t^{(1)}(\lambda)$. In relation (77), one can see that besides the wanted term which is dedicated to the eigenvalue, we also have the unwanted terms which must be canceled so that the assumed eigenvector is indeed the eigenvector of the transfer matrix. With the help of the symmetry property (66) of the assumed eigenvector (72), use the standard algebraic Bethe ansatz method, we find if $\mu_1, \cdots, \mu_n$ satisfy the following Bethe ansatz equations, the unwanted terms will vanish.

$$a^{-2L}(\mu_j)U_3(\mu_j)U_3^+(\mu_j)\frac{\mu_j}{\mu_j+i}\prod_{i=1,i\neq j}^{n}\frac{(\mu_j+\mu_i)(\mu_j-\mu_i-i)}{(\mu_j-\mu_i+i)(\mu_j+\mu_i+2i)} = \Lambda^{(1)}(\mu_j), \quad j=1,2,\cdots,n. \tag{85}$$

Here we have used the notation $\Lambda^{(1)}$ to denote the eigenvalue of the nested transfer matrix $t^{(1)}(\lambda)$.

Thus what we should do next is to find the eigenvalue of the nested transfer matrix $t^{(1)}$.

IV.3 The nested algebraic Bethe ansatz method

We hope that the eigenvalue of the nested transfer matrix can be solved similarly as that of the original transfer matrix. It seems that there is a logic error to call $t^{(1)}$ the transfer matrix, because that we did not show $t^{(1)}$ commute with each other for different spectral parameters. On the other hand, $K^{(1)}$ and $K^{(1)+}$ have already been defined explicitly in the above section, they are not obtained from, for example, reflection equation and the dual reflection equation. In this section, we will shown that all of those problems can be solved in the framework of the reflecting boundary condition case.

We know that the following graded Yang-Baxter relation with grading $\epsilon_1 = \epsilon_2 = 1$ is satisfied:

$$r(\lambda-\mu)L_1^{(1)}(\lambda)L_2^{(1)}(\mu) = L_2^{(1)}(\mu)L_1^{(1)}(\lambda)r(\lambda-\mu) \tag{86}$$

So, we also have the Yang-Baxter relation for the row-to-row monodromy matrix

$$r(\lambda-\mu)T_1^{(1)}(\lambda,\{\mu_i\})T_2^{(1)}(\mu,\{\mu_i\}) = T_2^{(1)}(\mu,\{\mu_i\})T_1^{(1)}(\lambda,\{\mu_i\})r(\lambda-\mu). \tag{87}$$

We propose the reflection equation take the form

$$\begin{aligned} & r_{12}(\lambda-\mu)K_1^{(1)}(\lambda)r_{21}(\lambda+\mu)K_2^{(1)}(\mu) \\ = \; & K_2^{(1)}(\mu)r_{12}(\lambda+\mu)K_1^{(1)}(\lambda)r_{21}(\lambda-\mu). \end{aligned} \tag{88}$$

Solving this reflection equation directly, we find

$$K^{(1)}(\lambda) = \begin{pmatrix} \xi+\lambda & \\ & \xi-\lambda \end{pmatrix} \tag{89}$$

is a solution of the reflection equation, it is easy to show that $K^{(1)} = id.$ is also a solution. These solutions of the reflection equation are consistent with the results defined in the above subsetion. So, we can show that the nested double-row monodromy matrix

$$\mathcal{T}^{(1)}(\lambda,\{\mu_i\}) \equiv T^{(1)}(\lambda,\{\mu_i\})K^{(1)}(\lambda){T^{(1)}}^{-1}(-\lambda,\{\mu_i\}) \tag{90}$$

also satisfy the the reflection equation,

$$\begin{aligned} & r_{12}(\lambda-\mu)\mathcal{T}_1^{(1)}(\lambda,\{\mu_i\})r_{21}(\lambda+\mu)\mathcal{T}_2^{(1)}(\mu,\{\mu_i\}) \\ = \; & \mathcal{T}_2^{(1)}(\nu,\{\mu_i\})r_{12}(\lambda+\mu)\mathcal{T}_1^{(1)}(\lambda,\{\mu_i\})r_{21}(\lambda-\mu). \end{aligned} \tag{91}$$

One can prove that the r-matrix satisfy a unitarity relation

$$r_{12}^{st_1}(\lambda)r_{21}^{st_1}(2i-\lambda) = a(\lambda)a(2i-\lambda)\cdot id. \tag{92}$$

According to this relation and the unitarity relation of the r-matrix $r_{12}(\lambda)r_{21}(-\lambda) = 1\cdot id.$, we propose the following dual reflection equation

$$\begin{aligned} & r_{12}(\mu-\lambda){K_1^{(1)}}^{+}(\lambda)r_{21}(\lambda+\mu+2i){K_2^{(1)}}^{+}(\mu) \\ = \; & {K_2^{(1)}}^{+}(\mu)r_{12}(\lambda+\mu+2i){K_1^{(1)}}^{+}(\lambda)r_{21}(\mu-\lambda). \end{aligned} \tag{93}$$

We can also find that the solution of the dual reflection equation is consistent with the ${K^{(1)}}^{+}$ results (80) presented in the above subsetion. Similar procedure as that presented in section III can also be applied now, we find that the defined nested transfer matrix indeed consitute a commuting family for different spectral parameters.

Now, let us use again the algebraic Bethe ansatz method to obtain the eigenvalue $\Lambda^{(1)}(\lambda)$ of the nested transfer matrix $t^{(1)}(\lambda)$. We write the nested double-row monodromy matrix as:

$$\begin{aligned} \mathcal{T}^{(1)}(\lambda,\{\mu_i\}) &= \begin{pmatrix} \mathcal{A}^{(1)}(\lambda) & \mathcal{B}^{(1)}(\lambda) \\ \mathcal{C}^{(1)}(\lambda) & \mathcal{D}^{(1)}(\lambda) \end{pmatrix} \\ &= T^{(1)}(\lambda,\{\mu_i\})K^{(1)}(\lambda){T^{(1)}}^{-1}(-\lambda,\{\mu_i\}) \\ &= \begin{pmatrix} A^{(1)}(\lambda) & B^{(1)}(\lambda) \\ C^{(1)}(\lambda) & D^{(1)}(\lambda) \end{pmatrix} \begin{pmatrix} k_1^{(1)}(\lambda) & \\ & k_2^{(1)}(\lambda) \end{pmatrix} \begin{pmatrix} \bar{A}^{(1)}(-\lambda) & \bar{B}^{(1)}(-\lambda) \\ \bar{C}^{(1)}(-\lambda) & \bar{D}^{(1)}(-\lambda) \end{pmatrix} \end{aligned} \tag{94}$$

For convenience, we introduce again a transformation

$$\mathcal{A}^{(1)}(\lambda) = \tilde{\mathcal{A}}^{(1)}(\lambda) - \frac{i}{2\lambda-i}\mathcal{D}^{(1)}(\lambda). \tag{95}$$

Because that the nested double-row monodromy matrix satisfy the reflection equation (91), we can find the following commutation relations:

$$\begin{aligned}\mathcal{D}^{(1)}(\lambda)\mathcal{C}^{(1)}(\mu) &= \frac{(\lambda-\mu+i)(\lambda+\mu)}{(\lambda-\mu)(\lambda+\mu-i)}\mathcal{C}^{(1)}(\mu)\mathcal{D}^{(1)}(\lambda) \\ &- \frac{2i\mu}{(\lambda-\mu)(2\mu-i)}\mathcal{C}^{(1)}(\lambda)\mathcal{D}^{(1)}(\mu)+\frac{i}{\lambda+\mu-i}\mathcal{C}^{(1)}(\lambda)\tilde{\mathcal{A}}^{(1)}(\mu), \quad (96)\\ \tilde{\mathcal{A}}^{(1)}(\lambda)\mathcal{C}^{(1)}(\mu) &= \frac{(\lambda-\mu-i)(\lambda+\mu-2i)}{(\lambda-\mu)(\lambda+\mu-i)}\mathcal{C}^{(1)}(\mu)\tilde{\mathcal{A}}^{(1)}(\lambda) \\ &+ \frac{2i(\lambda-i)}{(\lambda-\mu)(2\lambda-i)}\mathcal{C}^{(1)}(\lambda)\tilde{\mathcal{A}}^{(1)}(\mu)-\frac{4i\mu(\lambda-i)}{(\lambda+\mu-i)(2\lambda-i)(2\mu-i)}, \quad (97)\\ \mathcal{C}^{(1)}(\lambda)\mathcal{C}^{(1)}(\mu) &= \mathcal{C}^{(1)}(\mu)\mathcal{C}^{(1)}(\lambda). \quad (98)\end{aligned}$$

As the reference states, for the nesting we pick

$$|0>_k^{(1)}=\begin{pmatrix}0\\1\end{pmatrix}, |0>^{(1)}=\otimes_{k=1}^n|0>_k^{(1)}\ . \quad (99)$$

With the help of the definition (82, 84), we know the results of the nested monodromy matrix and the inverse of the monodromy matrix acting on the reference state

$$\begin{aligned}T^{(1)}(\lambda,\{\mu_i\})|0>^{(1)} &= \begin{pmatrix}A^{(1)}(\tilde{\lambda}) & B^{(1)}(\tilde{\lambda})\\ C^{(1)}(\tilde{\lambda}) & D^{(1)}(\tilde{\lambda})\end{pmatrix}|0>^{(1)}\\ &= \begin{pmatrix}\prod_{i=1}^n a(\tilde{\lambda}+\tilde{\mu}_i) & 0\\ C^{(1)}(\tilde{\lambda}) & \prod_{i=1}^n[a(\tilde{\lambda}+\tilde{\mu}_i)-b(\tilde{\lambda}+\tilde{\mu})]\end{pmatrix}|0>^{(1)},\end{aligned} \quad (100)$$

$$\begin{aligned}T^{(1)^{-1}}(-\lambda,\{\mu_i\})|0>^{(1)} &= \begin{pmatrix}\bar{A}^{(1)}(\tilde{\lambda}) & \bar{B}^{(1)}(\tilde{\lambda})\\ \bar{C}^{(1)}(\tilde{\lambda}) & \bar{D}^{(1)}(\tilde{\lambda})\end{pmatrix}|0>^{(1)}\\ &= \begin{pmatrix}\prod_{i=1}^n a(\tilde{\lambda}-\tilde{\mu}_i) & 0\\ C^{(1)}(\tilde{\lambda}) & \prod_{i=1}^n[a(\tilde{\lambda}-\tilde{\mu}_i)-b(\tilde{\lambda}-\tilde{\mu})]\end{pmatrix}|0>^{(1)}\ .\end{aligned} \quad (101)$$

Substituting those relation into relation (94), we can obtain the results of the nested double-row monodromy matrix acting on the nested vacuum state $|0>^{(1)}$,

$$\mathcal{B}^{(1)}(\tilde{\lambda})|0>^{(1)}=0, \quad \mathcal{C}^{(1)}(\tilde{\lambda})|0>^{(1)}\neq 0, \quad (102)$$

$$\mathcal{D}^{(1)}(\tilde{\lambda})|0>^{(1)}=U_2(\tilde{\lambda})\prod_{i=1}^n\left\{[a(\tilde{\lambda}+\tilde{\mu}_i)-b(\tilde{\lambda}+\tilde{\mu}_i)][a(\tilde{\lambda}-\tilde{\mu}_i)-b(\tilde{\lambda}-\tilde{\mu}_i)]\right\}|0>^{(1)}\ . \quad (103)$$

Here we use the notation $U_2 = k_2^{(1)}$, $U_2(\tilde{\lambda}) = \frac{(2\tilde{\lambda}-i)(\tilde{\lambda}+\tilde{\xi})}{2\tilde{\lambda}}$ for K_I case, $U_2(\tilde{\lambda}) = \frac{(2\tilde{\lambda}-i)(\tilde{\xi}-\tilde{\lambda})}{2\tilde{\lambda}}$ for K_{II} case.

The result of element $\tilde{A}^{(1)}$ is not so direct. We know

$$\mathcal{A}^{(1)}(\tilde{\lambda})|0>^{(1)} = k_1^{(1)}(\tilde{\lambda})A^{(1)}(\tilde{\lambda})\bar{A}^{(1)}(-\tilde{\lambda})|0>^{(1)} + k_2^{(1)}(\tilde{\lambda})B^{(1)}(\tilde{\lambda})\bar{C}^{(1)}(-\tilde{\lambda})|0>^{(1)} . \quad (104)$$

Using the Yang-Baxter relation (87), the above relation becomes

$$\begin{aligned}\mathcal{A}^{(1)}(\tilde{\lambda})|0>^{(1)} &= k_1^{(1)}(\tilde{\lambda})A^{(1)}(\tilde{\lambda})\bar{A}^{(1)}(-\tilde{\lambda})|0>^{(1)} \\ &+ k_2^{(1)}(\tilde{\lambda})\frac{b(2\tilde{\lambda})}{a(2\tilde{\lambda})-b(2\tilde{\lambda})}[A^{(1)}(\tilde{\lambda})\bar{A}^{(1)}(-\tilde{\lambda}) - \bar{D}^{(1)}(-\tilde{\lambda})D^{(1)}(\tilde{\lambda})]|0>^{(1)} \\ &= [k_1^{(1)}(\tilde{\lambda}) + k_2^{(1)}(\tilde{\lambda})\frac{b(2\tilde{\lambda})}{a(2\tilde{\lambda})-b(2\tilde{\lambda})}]\prod_{i=1}^{n}[a(\tilde{\lambda}+\tilde{\mu}_i)a(\tilde{\lambda}-\tilde{\mu}_i)]|0>^{(1)} \\ &- \frac{i}{2\tilde{\lambda}-i}\mathcal{D}^{(1)}(\tilde{\lambda})|0>^{(1)} . \end{aligned} \quad (105)$$

With the help of the transformation (94), we find

$$\tilde{\mathcal{A}}^{(1)}(\tilde{\lambda})|0>^{(1)} = U_1(\tilde{\lambda})\prod_{i=1}^{n}[a(\tilde{\lambda}+\tilde{\mu}_i)a(\tilde{\lambda}-\tilde{\mu}_i)]|0>^{(1)}, \quad (106)$$

where we denote $U_1(\tilde{\lambda}) = k_1^{(1)}(\tilde{\lambda}) + k_2^{(1)}(\tilde{\lambda})\frac{b(2\tilde{\lambda})}{a(2\tilde{\lambda})-b(2\tilde{\lambda})}$, and U_1 take the following form explicitly

For $K_I(\lambda)$:

$$U_1(\tilde{\lambda}) = \tilde{\lambda} + \tilde{\xi}, \quad (107)$$

For $K_{II}(\lambda)$:

$$U_1(\tilde{\lambda}) = \tilde{\lambda} + \tilde{\xi} - i. \quad (108)$$

The nested transfer matrix takes the form

$$\begin{aligned} t^{(1)}(\tilde{\lambda}) &= strK^{(1)}(\tilde{\lambda})\mathcal{T}^{(1)}(\tilde{\lambda}) \\ &= -k_1^{(1)+}(\tilde{\lambda})\mathcal{A}^{(1)}(\tilde{\lambda}) - k_2^{(1)+}(\tilde{\lambda})\mathcal{D}^{(1)}(\tilde{\lambda}) \\ &= -U_1^+(\tilde{\lambda})\tilde{\mathcal{A}}^{(1)}(\tilde{\lambda}) - U_2^+(\tilde{\lambda})\mathcal{D}^{(1)}(\tilde{\lambda}), \end{aligned} \quad (109)$$

where we denote $U_1^+ = k_1^{(1)+}$, $U_2^+(\lambda) = k_2^{(1)+}(\lambda) - \frac{i}{2\lambda - i}k_1^{(1)+}(\lambda)$, that means:

For K_I^+ case:

$$U_1^+(\tilde{\lambda}) = \tilde{\xi}^+ + i - \tilde{\lambda}, \quad (110)$$

$$U_2^+(\tilde{\lambda}) = \frac{2(\tilde{\lambda}-i)(\tilde{\xi}^+ + i - \tilde{\lambda})}{2\tilde{\lambda}-i}, \quad (111)$$

For K_{II}^+ case:

$$\begin{aligned} U_1^+(\tilde{\lambda}) &= \tilde{\xi}^+ + i - \tilde{\lambda}, \\ U_2^+(\tilde{\lambda}) &= \frac{2(\tilde{\lambda}+\tilde{\xi}^+)(\tilde{\lambda}-i)}{2\tilde{\lambda}-i}. \end{aligned} \tag{112}$$

Using the standard algebraic Bethe ansatz method, assume that the eigenvector of the nested transfer matrix constucted as $\mathcal{C}(\tilde{\mu}_1^{(1)})\mathcal{C}(\tilde{\mu}_2^{(1)})\cdots\mathcal{C}(\tilde{\mu}_m^{(1)})|0>^{(1)}$. Acting the nested transfer matrix on this eigenvector, using repeatedly the commutation relations (96,97), we have the eigenvalue

$$\begin{aligned} \Lambda^{(1)}(\tilde{\lambda}) &= -U_1^+(\tilde{\lambda})U_1(\tilde{\lambda})\prod_{i=1}^{n}[a(\tilde{\lambda}+\tilde{\mu}_i)a(\tilde{\lambda}-\tilde{\mu}_i)]\prod_{l=1}^{m}\left\{\frac{(\tilde{\lambda}-\tilde{\mu}_l^{(1)}+i)(\tilde{\lambda}+\tilde{\mu}_l^{(1)})}{(\tilde{\lambda}-\tilde{\mu}_l^{(1)})(\tilde{\lambda}+\tilde{\mu}_l^{(1)}-i)}\right\} \\ &\quad -U_2^+(\tilde{\lambda})U_2(\tilde{\lambda})\prod_{i=1}^{n}\Big\{[a(\tilde{\lambda}+\tilde{\mu}_i)-b(\tilde{\lambda}+\tilde{\mu}_i)][a(\tilde{\lambda}-\tilde{\mu}_i)-b(\tilde{\lambda}-\tilde{\mu}_i)]\Big\} \\ &\quad \prod_{l=1}^{m}\left\{\frac{(\tilde{\lambda}-\tilde{\mu}_l^{(1)}-i)(\tilde{\lambda}+\tilde{\mu}_l^{(1)}-2i)}{(\tilde{\lambda}-\tilde{\mu}_l^{(1)})(\tilde{\lambda}+\tilde{\mu}_l^{(1)}-i)}\right\}, \end{aligned} \tag{113}$$

where $\tilde{\mu}_1^{(1)},\cdots,\tilde{\mu}_m^{(1)}$ should satisfy the following Bethe ansatz equations

$$\begin{aligned} &\frac{U_1^+(\tilde{\mu}_j^{(1)})U_1(\tilde{\mu}_j^{(1)})}{U_2^+(\tilde{\mu}_j^{(1)})U_2(\tilde{\mu}_j^{(1)})}\frac{\tilde{\mu}_j^{(1)}}{(\tilde{\mu}_j^{(1)}-i)}\prod_{i=1}^{n}\frac{(\tilde{\mu}_j^{(1)}+\tilde{\mu}_i)(\tilde{\mu}_j^{(1)}-\tilde{\mu}_i)}{(\tilde{\mu}_j^{(1)}+\tilde{\mu}_i-i)(\tilde{\mu}_j^{(1)}-\tilde{\mu}_i-i)} \\ &= \prod_{l=1,\neq j}^{m}\frac{(\tilde{\mu}_j^{(1)}-\tilde{\mu}_l^{(1)}-i)(\tilde{\mu}_j^{(1)}+\tilde{\mu}_l^{(1)}-2i)}{\tilde{\mu}_j^{(1)}-\tilde{\mu}_l^{(1)}+i)(\tilde{\mu}_j^{(1)}+\tilde{\mu}_l^{(1)})}, \quad j=1,\cdots,m. \end{aligned} \tag{114}$$

Thus, the eigenvalue of the transfer matrix $t(\lambda)$ with reflecting boundary condition (36) is obtained as:

$$\begin{aligned} \Lambda(\lambda) &= U_3^+(\lambda)U_3(\lambda)\prod_{i=1}^{n}\frac{(\lambda+\mu_i)(\lambda-\mu_i-i)}{(\lambda+\mu_i+i)(\lambda-\mu_i)} \\ &\quad + a^{2L}(\lambda)\prod_{i=1}^{n}\frac{(\lambda-\mu_i+i)(\lambda+\mu_i+2i)}{(\lambda-\mu_i)(\lambda+\mu_i+i)}\Lambda^{(1)}(\tilde{\lambda}). \end{aligned} \tag{115}$$

Here for convenience, we give summary of the values U and U^+.

Case I:

$$\begin{aligned} U_1^+(\tilde{\lambda}) &= \tilde{\xi}^+ + i - \tilde{\lambda}, \\ U_2^+(\tilde{\lambda}) &= \frac{2(\tilde{\lambda}-i)(\tilde{\xi}^+ + i - \tilde{\lambda})}{2\tilde{\lambda}-i}, \\ U_3^+(\lambda) &= \frac{(2\lambda-i)(\xi^+ + \lambda + i)}{2\lambda+i}, \end{aligned} \tag{116}$$

Case II:

$$\begin{aligned} U_1^+(\tilde{\lambda}) &= \tilde{\xi}^+ + i - \tilde{\lambda}, \\ U_2^+(\tilde{\lambda}) &= \frac{2(\tilde{\lambda} + \tilde{\xi}^+)(\tilde{\lambda} - i)}{2\tilde{\lambda} - i}. \\ U_3^+(\lambda) &= \frac{(2\lambda - i)(\xi^+ + \lambda)}{2\lambda + i}. \end{aligned} \tag{117}$$

Case I:

$$\begin{aligned} U_1(\tilde{\lambda}) &= \tilde{\lambda} + \tilde{\xi}, \\ U_2(\tilde{\lambda}) &= \frac{(2\tilde{\lambda} - i)(\tilde{\lambda} + \tilde{\xi})}{2\tilde{\lambda}}, \\ U_3(\lambda) &= (\xi - \lambda) \end{aligned} \tag{118}$$

Case II:

$$\begin{aligned} U_1(\tilde{\lambda}) &= \tilde{\lambda} + \tilde{\xi} - i. \\ U_2(\tilde{\lambda}) &= \frac{(2\tilde{\lambda} - i)(\tilde{\xi} - \tilde{\lambda})}{2\tilde{\lambda}}, \\ U_3(\lambda) &= (\xi - \lambda). \end{aligned} \tag{119}$$

We know that U and U^+ are independent of each other, so there are four combinations for $\{U, U^+\}$ such as $\{I, I\}$, $\{I, II\}$, $\{II, I\}$ and $\{II, II\}$.

V Summary and discussions

In this paper, we study the supersymmetric $t-J$ model with reflecting boundary conditions. We first studied the unitarity relation and the cross-unitarity relation of the R matrix. According those relations, we proposed the reflection equation and the dual reflection equation for this supersymmetric $t-J$ model. Solving the reflection equation and to dual reflection equation we give two types solutions for them respectively. The transfer matrix for the supersymmetric $t-J$ model is then constructed, and we proved that the transfer matrix constitute a commuting family. Using the nested algebraic Bethe ansatz method, we obtain the eigenvalues of the transfer matrix.

We discussed all of the above in the FFB grading. For the periodic boundary conditions, the supersymmetric $t-J$ model was studied in three different background grading.[3] We can also study the FBF and BFF grading for the supersymmetric $t-J$ model with reflecting boundary conditions. The integrability can be proved similar as that of in this paper. We have found the unitarity and cross-unitarity relations of the R matrix take the

same form as that of FFB grading, we also have a same solutions for the reflection equation. Using similar nested algebraci Bethe ansatz method, we can find the eigenvalues of the transfer matrix with reflecting boundary conditions with grading FBF and BFF.

The R-matrix of the supersymmetric $t-J$ model are rational R-matrix, there are trigonometric R-matrix corresponding to a generalized supersymmetric $t-J$ model. We can also study this generalized supersymmetric $t-J$ model with reflecting boundary conditions.

It is interest to continue study the thermodynamic limit of the result obtained in this paper. Thus we can find some physical quantites such as free energy, surface free energy and interfacial tension ect..

We can also extend the supersymmetric $t-J$ model to a more general supersymmetric case. The R-matrix will equivalent to the R-matrix of the Perk-Shultz model.[12]

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